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LETTER TO THE EDITOR

A Parisi equation for Sompolinsky's solution of the SK model

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Abstract. Using a replica formalism, we show how Sompolinsky's free energy functional for the SK model of spin glasses can be expressed as the solution of a differential equation; as a special case, one recovers Parisi's equation.

The SK model of a spin glass (Sherrington and Kirkpatrick 1975) has recently been considered from a dynamical point of view by Sompolinsky (1981). His description involves a double continuum of order parameters, $q(x)$ and $\Delta'(x)$, and contains the salient features of Parisi's solution (Parisi 1980a and references therein). Subsequently, DGO (de Dominicis *et al* 1981) have shown how Sompolinsky's free energy functional can be recovered on a static basis, using the replica method.

It is the purpose of the present Letter to write this free energy in terms of the solution of a differential equation subject to a certain boundary condition. Further, on setting $\Delta'(x) = -xq'$, Parisi's differential equation (Parisi 1980b, Duplantier 1981) is recovered.

In a field h, the free energy functional per spin of the SK model reads

$$-\frac{f}{T}=\frac{J^2}{4T^2}+\lim_{n\to 0}\frac{1}{n}\max\left[-\frac{J^2}{4T^2}\sum_{\alpha\neq\beta}q_{\alpha\beta}^2+\ln\operatorname*{Tr}_{(n)}\exp\left(\frac{J^2}{2T^2}\sum_{\alpha\neq\beta}q_{\alpha\beta}\sigma_\alpha\sigma_\beta+\frac{h}{T}\sum_\alpha\sigma_\alpha\right)\right] \quad (1)$$

where $\mathrm{Tr}_{(n)}$ means the trace over n replicas. In the DGO iteration scheme, the $q_{\alpha\beta}$ matrix is initially divided into n/p_0 diagonal $p_0\times p_0$ blocks (of elements q_0) and $(n/p_0)(n/p_0-1)$ off-diagonal blocks (of elements r_0). The transformation consists in applying a Parisi-like scheme on both q_0 and r_0 blocks.

Using this transformation K times, with block sizes $p_0, p_1, \ldots p_K$ yields

$$-\frac{f}{T}=\frac{J^2}{4T^2}\left((q_K-1)^2+2\sum_{l=0}^{K}q_l\Delta_l'\right)+\lim_{\substack{n\to 0\\ p_0\gg p_1\gg\ldots\gg p_K\to\infty}}G_{-1}(h) \quad (2)$$

where $p_0, p_1, \ldots, p_K$ go to infinity successively and in order, and with

$$\exp nG_{-1}(h)=\operatorname*{Tr}_{(n)}\exp\left(\frac{J^2}{2T^2}\sum_{\alpha\neq\beta}q_{\alpha\beta}\sigma_\alpha\sigma_\beta+\frac{J^2}{2T^2}nq_K+\frac{h}{T}\sum_\alpha\sigma_\alpha\right)$$

$$=\int\frac{\mathrm{d}z_0}{\sqrt{2\pi}}\exp-\frac{z_0^2}{z}\prod_{j_1}\frac{\mathrm{d}z_{j_1}}{\sqrt{2\pi}}\exp-\tfrac{1}{2}z_{j_1}^2\ldots\prod_{j_1\ldots j_K}\exp-\tfrac{1}{2}z_{j_1\ldots j_K}^2$$

$$\times \prod_{j_0=1}^{n/p_0} \left[\!\left[\int \frac{\mathrm{d}y_{j_0}}{\sqrt{2\pi}} \exp - p_0 \tfrac{1}{2} y_{j_0}^2 \prod_{j_1=1}^{p_0/p_1} \left(\int \frac{\mathrm{d}y_{j_0 j_1}}{\sqrt{2\pi}} \exp - p_1 \tfrac{1}{2} y_{j_0 j_1}^2 \right.\right.\right.$$

$$\times \prod_{j_2=1}^{p_1/p_2} \left\{ \dots \prod_{j_K=1}^{p_{K-1}/p_K} \left[\int \frac{\mathrm{d}y_{j_0 \dots j_K}}{\sqrt{2\pi}} \exp - p_K \tfrac{1}{2} y_{j_0 \dots j_K}^2 \right.\right.$$

$$\times \exp p_K \ln 2 \cosh \left(\frac{h}{T} + \frac{J}{T} q_0^{1/2} z_0 + \dots + \frac{J}{T} (q_K - q_{K-1})^{1/2} z_{j_1 \dots j_K} \right.$$

$$\left.\left.\left.\left.\left. + \frac{J}{T} (-\Delta_0')^{1/2} y_{j_1} + \dots + \frac{J}{T} (-\Delta_K')^{1/2} y_{j_0 \dots j_K} \right) \right] \dots \right\} \right) \right]\!\right] \qquad (3)$$

where we have dropped p factors that do not contribute in the limit $n \to 0$, $p_0 \gg p_1 \gg \dots \gg p_K \to \infty$.

Following Parisi this structure suggests introducing quantities G_l such that

$$\exp p_{l-1} G_{l-1}(h)$$

$$= \int \frac{\mathrm{d}z_l}{\sqrt{2\pi}} \exp - \tfrac{1}{2} z_l^2$$

$$\times \left(\int \frac{\mathrm{d}y_l}{\sqrt{2\pi}} \exp p_l [- \tfrac{1}{2} y_l^2 + G_l (h + J\sqrt{q_l - q_{l-1}} z_l + J\sqrt{-\Delta_l'} y_l)] \right)^{p_{l-1}/p_l}$$

$$l = 0, 1, \dots K \qquad p_{-1} \equiv n \qquad q_{-1} \equiv o \qquad (4)$$

with

$$\exp p_l G_l (h + J\sqrt{q_l - q_{l-1}} z_l + J\sqrt{-\Delta_l'} y_l)$$

$$= \operatorname*{Tr}_{(p_l)} \exp \left\{ \left[\sum_{i=0}^{l} \left(\frac{J}{T} \sqrt{q_i - q_{i-1}} z_i + \frac{J}{T} \sqrt{-\Delta_i'} y_i \right) + \frac{h}{T} \right] \right.$$

$$\left. \times \sum_{\alpha=1}^{p_l} \sigma_l + \frac{J^2}{2T^2} \sum_{\substack{\alpha \neq \beta \\ 1}}^{p_l} (q_{\alpha\beta} \, (q_{\alpha\beta} - q_l) \sigma_\alpha \sigma_\beta \right\} \qquad (5)$$

(y_l, z_l stand for $y_{j_0 \dots j_l}$ and $z_{j_1 \dots j_l}$ respectively). Note that the block structure appearing in DGO is recovered by writing the last term in (5) as

$$\sum_{\alpha \neq \beta}^{p_l} (r_{\alpha\beta} - r_l) \sigma_\alpha \sigma_\beta + \sum_{\alpha \neq \beta}^{p_l} [(q_{\alpha\beta} - q_l) - (r_{\alpha\beta} - r_l)] \sigma_\alpha \sigma_\beta$$

where the diagonal ($q_{\alpha\beta}$) and off-diagonal ($r_{\alpha\beta}$) blocks are now exhibited. In the limit $p_l \to \infty$, a saddle point method on the y_l variable of equation (4) yields

$$y_c = \partial G_l / \partial y_c \qquad (6a)$$

$$\exp p_{l-1} G_{l-1}(h) = \int \frac{\mathrm{d}z_l}{\sqrt{2\pi}} \exp - \tfrac{1}{2} z_l^2$$

$$\times \exp p_{l-1} \left\{ - \tfrac{1}{2} y_c^2 + G_l (h + J\sqrt{q_l - q_{l-1}} z_l + J\sqrt{-\Delta_l'} y_c) \right\}. \qquad (6b)$$

As $K \to \infty$, $l/K \to x$, $G_l(h) \to G(x, h)$ and, to order δx, $q_l - q_{l-1} \to q'(x) \delta x$,

$$\Delta_l' \to \Delta'(x) \delta x \qquad y_c \to J(-\Delta'(x)\delta x)^{1/2} (\partial G / \partial h)$$

$$G_l \to G(x, h) + J\sqrt{q'(x)\delta x} z (\partial G / \partial h) + \tfrac{1}{2} J^2 z^2 q'(x) \delta x (\partial^2 G / \partial h^2) - J^2 \Delta'(x) \delta x (\partial G / \partial h)^2.$$

Inserting these expressions in (6*b*) and identifying the coefficients of the expansion in powers of p_{l-1} of the left- and right-hand side we finally obtain the desired differential equation

$$\frac{\partial G}{\partial x}(x, h) = -\frac{J^2}{2}\left\{-\Delta'(x)\left(\frac{\partial G}{\partial h}\right)^2 + q'(x)\frac{\partial^2 G}{\partial h^2}\right]. \tag{7}$$

From (5) it is clear that the boundary condition on (7) is the same as Parisi's

$$G(1, h) = \ln(2\cosh h/T). \tag{8}$$

Setting $l = 0$ in (4) and taking the limit $K \to \infty$, the free energy (2) becomes:

$$-\frac{f}{T} = \frac{J^2}{4T^2}\left\{(1-q(1))^2 + 2\int_0^1 \Delta'(x)q(x)\,\mathrm{d}x\right\}$$
$$+\int \frac{\mathrm{d}z}{\sqrt{2\pi}}(\exp -\tfrac{1}{2}z^2)\left\{-\tfrac{1}{2}\left(\frac{\partial G}{\partial y_c}\right)^2 + G\left(0, h + Jz\sqrt{q(0)} + J\sqrt{-\Delta'(0)}\frac{\partial G}{\partial y_c}\right)\right] \tag{9}$$

$$y_c = \frac{\partial G}{\partial y_c}(0, h + Jz\sqrt{q(0)} + J\sqrt{-\Delta'(0)}y_c).$$

If we let $\Delta'(x) = -xq'(x)$, it is easily seen that (7) and (9) reduce to Parisi's form.

Note that, defining $u(x) = -\Delta'(x)/q'(x)$, *provided* $u(x)$ *is monotonic*, (7) reduces to

$$\frac{\partial G}{\partial u}(u, h) = -\tfrac{1}{2}J^2 q'(u)\left[u\left(\frac{\partial G}{\partial h}\right)^2 + \frac{\partial^2 G}{\partial h^2}\right] \tag{10}$$

which is the standard form of Parisi's equation (Parisi 1980b). However, the free energy (9) and equation (10) become identical to Parisi's results *only in the case when* $u(0)$ and $u(1) = 1$.

This is more general than any standard reparametrisation in Parisi's case ($x \to m(x)$, $m(x)$ arbitrary monotonic function) which, as one can easily check, leaves boundaries [0, 1], differential equation and free energy invariant.

References

de Dominicis C, Gabay M and Orland H 1981 *Saclay preprint*
Duplantier B 1981 *J. Phys. A: Math. Gen.* **14** 283
Parisi G 1980a *Phys. Rep.* **67** 25
—— 1980b *J. Phys. A: Math. Gen.* **13** L115
Sherrington D and Kirkpatrick D 1975 *Phys. Rev. Lett.* **35** 1792
Sompolinsky H 1981 *Phys. Rev. Lett.* **47** 935